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The effects of covariational reasoning on conceptual learning of functions with digital-enhanced experiments

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Covariation is a major aspect of functional thinking, but not easily accessible for students. Even more it is somewhat underrepresented in mathematics teaching at school compared to the correspondence aspect. Since experiments have proven to be particularly beneficial for functional thinking, two experiment-based approaches have been developed and evaluated with regards to the learning of functional thinking. One approach uses a numerical setting, which emphasises the correspondence aspect. A second, qualitative approach sets the focus on covariation. In this paper we discuss a qualitative analysis on the development of the covariation aspect during intervention in the numerical and the qualitative setting respectively in eight focus groups (high-/low-achieving), using the levels of covariational reasoning. In accordance with quantitative findings the groups in the qualitative setting show higher levels of covariational reasoning and more consistently dynamic argumentations.

Keywords: Covariation, functions, technology-enhanced learning, qualitative content analysis.

Conceptual learning of functions

Functional relationships count as one of the guiding ideas of mathematics education, many everyday phenomena can be structured with functions. To recognize functional relationships and describe them mathematically it requires functional thinking (FT). Vollrath (1989) describes FT with three aspects characteristic when dealing with functions. The first aspect is correspondence, where each value of an independent variable is assigned to exactly one value of a dependent variable. Functions are thus primarily static, with a local focus, as an correspondence rule of single values. The covariation aspect describes how two quantities co-vary, i.e. how the systematic variation of an independent variable results in the variation of the dependent variable. This emphasizes the dynamic component of FT. Vollrath's (1989) object aspect considers a functional relationship as a whole. It no longer focuses only on individual pairs of values or sections of the function, but the set of all value pairs. The function is understood as an independent object with properties and with which one can operate.

Despite its wide presence in mathematics curriculum, many students have difficulties in dealing with functions (Ganter, 2013). Although covariation is regarded as central to build adequate mental representations of functional contexts (Ruchniewicz, 2022), deficits can be found especially with the covariation aspect and at school the focus often lies on the correspondence aspect (ibid.). One reason of this imbalance is seen in a numerical introduction to functions, followed by the correspondence-based definition from Dirichlet, that induce a static view on functional relationships. A dynamic perspective on functions, prerequisite for the covariation aspect (Johnson, 2015), is postponed to high school mathematics. Or as Thompson and Carlson (2017) put it, there is a lack of opportunity and ability to reason dynamically for covariation.

Levels of Covariational Reasoning

Before developing a concept of covariation learners may only consider variation in a single variable (Johnson, 2015). The normative goal of the covariation conception is that the continuous variation of two variables is considered simultaneously (Thompson & Carlson, 2017). Carlson et al. (2002) describe five levels of competence in covariational reasoning in their covariation framework. Thompson and Carlson (2017) extend this by a sixth level. Covariational reasoning comprises of all cognitive activities of learners that involve variation of two variables in relation to each other (Carlson et al., 2002), including consideration of a pair of values, i.e., the correspondence aspect of FT. The levels of Carlson et al. (2002) are not described as consecutive stages in the learning process, but rather help to describe different conceptual pronunciation of covariation when solving specific tasks. Nevertheless, there is a hierarchy of the stages assumed, with a higher stage being associated with a more sophisticated concept of covariation (Carlson et al., 2002; Thompson & Carlson, 2017).

1. In Thompson and Carlson's (2017) first level (*no coordination*) learners have no conception of quantities varying together. They focus only on the unilateral variation of the dependent or the independent variable respectively. 2. With a *precoordination of values*, students consider an asynchronous covariation of variables: The change in one variable can already be attributed to the change in the other, but in a consecutive way: first one variable is changed and only afterwards the second one. 3. The *gross coordination of values* stage can be understood as directional covariation. Learners recognize the directions of the joint variation, (increase/decrease). 4. In the stage *coordination of values* the values of the independent and dependent variables are connected by forming discrete pairs of values (x, y). Since this stage is described as the correspondence aspect of FT, we use the *Quantitative Coordination* by Carlson et al. (2002) as fourth stage in our analysis, where students quantify the variation in both variables from one state to another. For example, at this level students would recognize that the circumference of a circle increases by about 3.14 cm when the diameter increases by 1 cm. Finally, Thompson and Carlson (2017) distinguish two types of continuous covariation in the last two stages: 5. while on the level of *chunky continuous covariation*, continuity is considered as covariation in chunks, i.e. within closed intervals, on 6. the *smooth continuous covariation* level the whole functional relationship is considered as continuous covariation. The exploration of functional relationships assumably promotes higher levels of covariation conception (Thompson & Carlson, 2017; Ruchniewicz, 2022).

Develop functional thinking with experiments

Numerical approaches induce a static view of functions and contradict the intuitive description of functional relationships between two variables with their joint variation (Johnson, 2015; Thompson & Carlson, 2017). Therefore, it seems promising to follow a qualitative approach with a focus on covariation (Thompson, 1994). As early as in primary school opportunities for experiencing functional relationships and covariation should be provided. In the lower secondary level, functions should then be used as descriptive means for real-world contexts (Ruchniewicz, 2022; Vollrath, 1989). Functional thinking is recognizable in the fact that students are able to use, interpret and translate between different forms of representation. Hence learning environments should promote representational switch (Lichti, 2019). According to Vollrath (1989) the ability to form hypotheses about the kind of relationship and about the influence of variation, to verify and revise them should be integrated in learning on FT, cognitive activities that also occur when experimenting. To identify, vary and observe the independent and dependent variable are also common for experimenting and for FT, which explains why experiment-based approaches have proven to be beneficial for FT (Ganter, 2013). When experimenting with hands-on material students can experience functional relationships immediately, which increases motivation and urges the learners to translate between reality and mathematics. On the other hand, it is demanding to deal with measurement errors or inaccurate data and the measurement procedure itself binds cognitive resources. Experimenting with simulations facilitates the experimental setup and measurement and gives the opportunity for didactic reduction. This enables systematic variation and observing covariation. When used as multi-representation system with an interactive digital experiment and for example a dynamic graph of the measurement data, it supports representational switch. Lichti (2019) reports, that simulations foster FT significantly better than hands-on material, but that both foster FT in different ways. Measurement procedures with hands-on material induce a static perspective for values and conditions, fostering the correspondence aspect. It stimulates basic modelling activities, relating the situation to mathematical description, while a simulation already contains a model of the situation. Simulations in turn, when used as multi-representational systems, illustrate connections between model and mathematical representations (e.g. graph and table) that evoke activities for representations and their transfer. The systematic variation in simulations develops a dynamic view, concerned with variation as well as transition and hence supports the covariation aspect.

The learning environments

The intervention analysed in this study is a self-guided hybrid learning environment as introduction to functional relationships. It is embedded in a story of two kids building a treehouse. With a workbook, a help guide, an experiment box and a website with simulations the learners are guided through three contexts with experiments, outlined in a hybrid way, with hands-on material and digital simulations. The contexts represent a linear, a quadratic and a varying functional relationship.

Two different settings are developed, a numerical and a qualitative setting, both using the same hands-on material, simulations and similar tasks. The students are grouped in four, two students working together as pair, the pairs working on similar contexts (see Figure 1) with hands-on material and simulations. In the numerical settings the experiments are outlined as follows: in the pre-experimental phase students make assumptions about the relationship, in the experimental phase they make measurements with the hands-on material and in the post-experimental phases they represent and interpret the data in the simulation, set their findings back into the context and abstract their findings through comparison with the related pair. In contrast, in the qualitative setting, a dynamic perspective is already in focus from the beginning and to quantify the functional relationship is not the objective. In the pre-experimental phase, the students estimate values of the dependent variable as a sequence (pattern) to predefined values of the independent. Instead of measuring on hands-on material, they use the simulation to carry out a digital experiment, observe the variation and covariation of the variables and describe it verbally. They interpret the graph in the post-experiment phase. After this shorter experimentation process, the whole group of four compare their contexts and findings to identify similarities. After this exchange phase, each pair carries out measurements on the context of the respective pair and verify their results. In a second group task, computational results are generated for each context. For further details on the learning environments see Digel and Roth (2022).

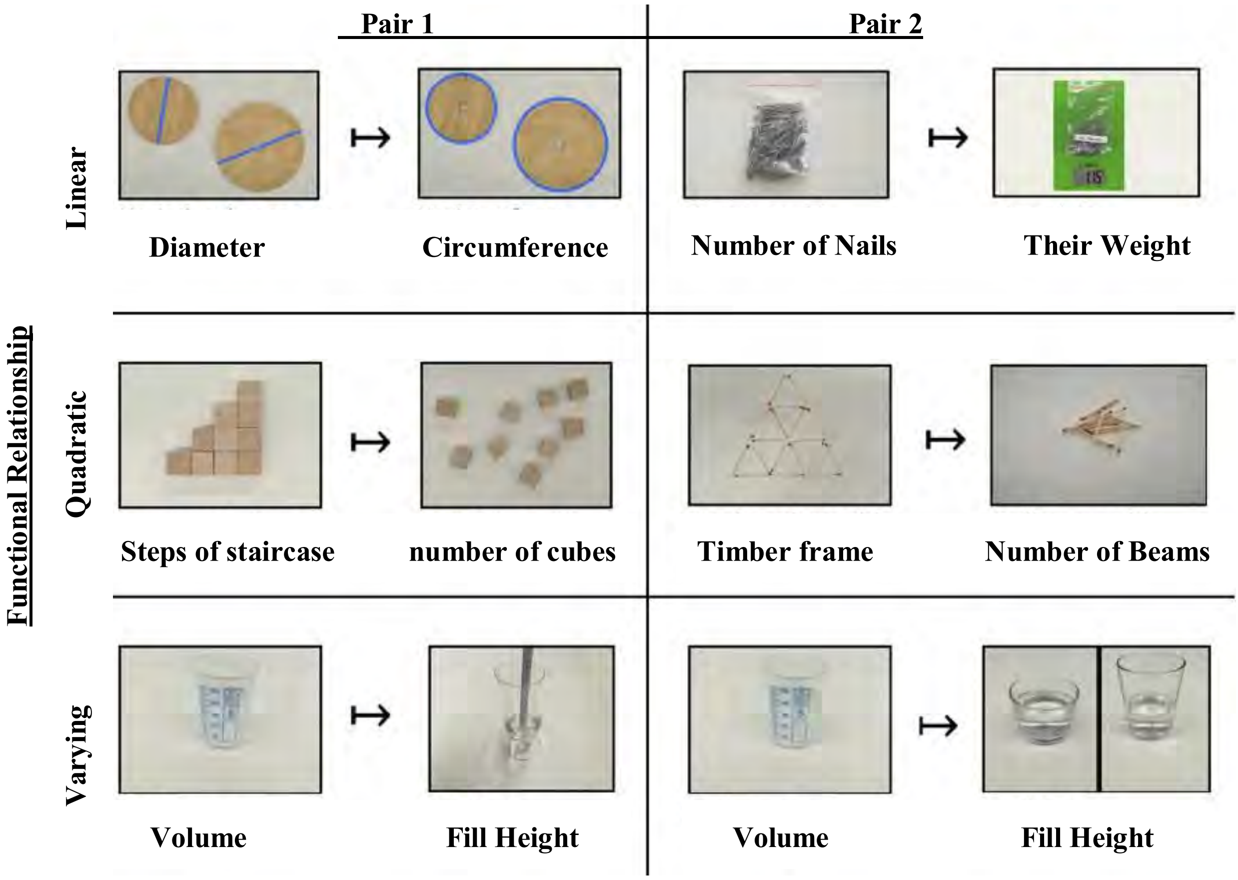

Figure 1: Contexts of the complementary pairs

Study design

The aim of the hybrid learning environment is to foster students' functional thinking and previous quantitative studies have shown, that the qualitative settings lead to significantly higher learning gains in FT than the numerical setting (Digel & Roth, 2022). According to Johnson (2015), the more opportunities are given to talk about covariation, the better it develops. These opportunities were explicitly created in the qualitative setting. In this study the following questions guide the analysis:

(RQ 1) How does the aspect of covariation develop during training?

(RQ 2) Are there differences in the development between the qualitative and the numerical approach?

16 high- and 16 low-performing students in the 6th and 7th grades of two grammar schools were videotaped when participating in the intervention with the numerical and the qualitative learning environments at the mathematical school lab of the University in Landau. Participation took place in July and August 2021. They worked in groups of four three times 90 minutes on one morning. All had little or no previous experience with functions in mathematics lessons. Two groups of four of the low-achievers and two groups of four of the high achievers worked on the numerical setting, the other two groups of low-achievers and high-achievers worked on the qualitative setting. Teachers classified the students as high-/low-performers.

Methods

To be able to assess how and in which tasks students developed a concept of covariation qualitative methods are used due to their open-ended, hypothesis-generating and theory-building nature. The statements made by students in group discussions are used to infer the cognitive processes and underlying concepts. Video, audio and screen recordings as well as the completed workbooks served as data. The versatile documentation of the process makes it possible to understand the process in retrospect. The transcription was done according to simple transcription rules (Kuckartz & Rädiker, 2022) in order to focus on the content. The contributions of students were marked with "S1:" to "S4:", the language was smoothed slightly, and dialects were not transcribed. In addition, actions performed on the hands-on material or simulations were also transcribed to be able to follow the statements (e.g.: S1 measures the diameter of the disc with the ruler). The data material consists of about 15 hours of video material as well as 165 pages of transcripts, which makes a highly reduced approach recommendable.

Describing and evaluating Covariational Reasoning

To examine the concept of covariation on which students argue at what point, content-structuring content analysis according to Kuckartz with seven phases is used. The rule-governed procedure minimizes the subjectivity of the researcher and make the procedure comprehensible and transparent (Kuckartz & Rädiker, 2022), leading to a differentiated system of categories. First, memos were written in phase 1 as part of the initiating text work. In the second phase correspondence, covariation and object were used as main categories and about 25% of the text material was coded, although some passages relevant to the research questions could not be assigned to any of the main categories. These passages coded as "other" were compared with each other, which resulted in three further main categories (unilateral variation; numerical solution methods; interpreting graphical representations). After all the material was coded with these six main categories in phase 3, phase 4 followed with the partly inductive, partly deductive formation of subcategories. The main category correspondence is divided inductively into the two subcategories static and dynamic. Five subcategories emerged for the main category of covariation. The literature comparison showed that the content of the inductively found categories corresponded to those of Ruchniewicz (2022), these designations and definitions were adopted and complemented with anchor examples from the available transcripts. This resulted in the subcategories asynchronous, directional, quantified, chunky and continuous covariation. Since the main category of object aspect was generally hardly evident in the students' argumentation, no subcategories were formed. The subcategories of the numerical solution methods and the interpretation of the graphical representation were chosen deductively. The subcategories for graphical representation were deductively adopted from Ruchniewicz (2022). Following the formation of the subcategories, in phase 5 the text passages previously coded with main categories were each assigned a subcategory which led to minor adaptations. The final coding manual can be downloaded in the GeoGebra learning environment (https://www.geogebra.org/m/nrvgxawv). In phase 6, the results were processed to enable interpretations. These results were interpreted in phase 7 to answer the research questions and generate hypotheses.

Results

The coding of all transcripts of the two pairs of low-achievers and two pairs of high-achievers resulted in the following frequencies of coding units of the main categories (table 1).

Table 1: Coded units in the main categories

	Numerical Setting				Qualitative Setting			
	High-achievers		Low-achievers		High-achievers		Low-achievers	
	Pair 1	Pair 2	Pair 1	Pair 2	Pair 1	Pair 2	Pair 1	Pair 2
Total units	82	89	26	33	56	175	52	60
Unilateral variation	6	10	3	5	8	33	9	10
Correspondence	24	31	14	11	12	38	14	24
Covariation	19	26	4	11	19	82	21	24
Object	-	-	-	1	-	-	-	-
Interpret graphical representation	25	22	-	5	11	15	6	1
Numerical solution	8	1	5	-	7	7	2	1

As a first result it is noticeable that in both groups of the numerical setting significantly less units were coded than in the groups of the qualitative setting (during the same time frame of intervention), i.e. students in the numerical settings discussed less content relevant to FT. Furthermore, students in the qualitative setting speak considerably more about covariation but not less about correspondence, in total and in relation to all coded units. Table 2 shows a timeline with coding units assigned to the subcategories related to FT resolved in the single tasks (1.1 – 4) of the learning environment in the groups of high-achievers. In the qualitative setting (top) static correspondence (sZ, yellow) still plays a major role in task 1, but hardly occurs in task 2 and already in task 1.4 it is superseded by quantified covariation (qKV, green). The focus of the argumentation of this group lies on covariation (green), it is constantly present across the tasks, develops in a staircase and there is a progression in the level of covariation (light green to dark green). Unilateral variation (orange) is present in many tasks, mostly of independent as well as dependent variable. The low-achievers qualitative group shows a similar, but weaker pattern for correspondence and covariation, with less units coded in total and more gaps. In the high-achievers numerical setting (table 2 bottom) it strikes that in total only few units were coded and there were no arguments present related to FT over several consecutive tasks. Starting with static correspondence, covariation first occurs in task 1.11 and is not taken up in longer passages, but only from task 2.8 on, where a progression takes place. Unilateral variation is only applied on the independent variable. The timeline of the low-achieving numerical group shows FT arguments only punctually (in six tasks), no progression and a low level of correspondence as well as covariation. Both high-achievers groups reach the level of continuous covariation. In tasks where covariation develops, the focus is on the representation forms simulation (S) and/or graph (G). It can thus be

Table 2: High-achievers' timeline of coded subcategories qualitative (top) / numerical (bottom) group

	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.10	1.11	1.12	1.13	1.14	1.15	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8
nLFV											2					1							
nLVG							1S(S)+G																
nLDS															3								
GÄS																							
GGS																							
GAS																							
Gb								2G			9			3G	1								
SG																							
kKV							1S(S)+G								1								
sKV							1S(S)+G				1												
qKV				8S(S)	2S(S)	9S(S)+(T)	4S(S)+G		2S(S)+G		7	2T	1T		5			2S(S)	4S(S)	1			
dKV							18S(S)+G	1G	3S(S)+G		2										2	2T+G	2G
aKV											1												
eVaV						2S(S)+(T)	4S(S)+G	1G			3					2T	2S(S)		1S(S)			1T+G	
eVuV						3S(S)+(T)	4S(S)+G	1G			2	7T											
dZ											1												
sZ		6S(S)	3S(S)			15S(S)+(T)	11S(S)+G				9	6T					1S(S)						
	nr	T	S(S)	S(S)	S(S)	S(S)+(T)	S(S)+G	G	S(S)+G	nr	A	T	T	G	A	T	S(S)	S(S)	S(S)	-	-	T+G	G

	1.1-1.2	1.3	1.4	1.5	1.6	1.7	1.8-1.9	1.10	1.11	1.12	1.13	1.14	2.1-2.6	2.7	2.8	2.9	2.10	2.11	2.12	3.1	3.2	3.3-3.6	3.7	3.8	3.9	3.10	3.11	4
nLFV						1T																						
nLVG												3																
nLDS												4																
GÄS																												
GGS																												
GAS											3G																	
Gb								2T+G	1S(S)+G	1G				2G	16G													
SG																												
kKV																									1G			1
sKV																								3G	2G			
qKV																	3T											1
dKV									2S(S)+G	1G		1																3
aKV									1S(S)+G																			
eVaV																												
eVuV								1T+G	2S(S)+G														3G					
dZ																					1S(M)							
sZ		3S(M)	3S(M)	10T								2						1	3		1S(M)							
	nr	S(M)	S(M)	T	nr	T	nr	T+G	S(S)+G	G	G	A	nr	G	G	nr	T	A	A	nr	S(M)	nr	G	G	G	A	A	-

confirmed that especially dynamic forms of representation promote the covariation aspect and thus also the development of FT. Covariation seems to play an essential role in translation processes between situation and graph and turns out to be the dominant notion.

Discussion

The multiple fixations of the data (video, audio, screen records, workbooks) results in a high level of internal validity (Kuckartz & Raediker, 2022). Although sample size is limited, the results are in line with quantitative results (Digel & Roth, 2022) and theory outlined earlier in this paper, showing external validity. In both the numerical and qualitative settings, the groups first argue based on the correspondence aspect before adopting the covariation perspective (RQ1), which indicates that the latter is not easy to grasp. Tasks that use situation with simulation and graph support the development of covariation for all high-achievers but only in the qualitative setting for low-achievers (RQ1). All high-achievers and the low-achievers in the qualitative setting argue over more than one level of covariation (RQ1). In all qualitative settings they tend to show staircases and a roughly chronological sequence along the levels according to Thompson and Carlson (2017), but as described by Carlson et al. (2002) not all stages are passed through consecutively. Exchange tasks support covariational argumentation and correspondence as well as covariation are intensely used. Thus, it can be concluded that especially with a qualitative approach to functions, the possibility of free exchange fosters covariation and FT. Covariation develops in the data from the lower to the higher levels of covariational reasoning. More exploration of functional relationships seems to help students reach higher levels of covariation understanding. Regarding unilateral variation (RQ2) the focus of the independent variable of high- and low-achievers numerical settings opposed to the diverse view on variation of independent and dependent variable in both qualitative settings (high/low) also confirms that the qualitative, dynamic approach is beneficial for covariational reasoning. Finally, there is a dominance of quantified covariation, also in the high-achiever groups. Arguing at this level of covariational reasoning seems to be the most accessible to the students, also in the qualitative groups.

To sum up, the results presented here confirm quantitative results of higher learning gains regarding functional thinking in the qualitative setting and stress the importance of covariation. The exchange tasks are identified as key phases, in which covariation is particularly promoted, which is in accordance with Thompson and Charlson's (2017) opportunity for covariational reasoning.

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