A qualitative-experimental approach to functional thinking with a focus on covariation

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Students encounter functional relationships in almost every grade. Nonetheless, they often experience difficulties when dealing with functions and show misconceptions. The prevalent numerical approaches to the topic in school practice lead to a pointwise view of functions which contributes to these problems. Experimental approaches have proven to be beneficial for functional thinking, with simulations inducing greater learning gains than experiments with real material. A closer look at these two methods reveals that each of them promotes a different aspect of functional thinking. The study presented here deals with the question of how both methods can be combined beneficially and proposes two different approaches.

Keywords: functional thinking, experiments, simulations, covariation.

FOSTERING FUNCTIONAL THINKING

According to Vollrath (1989), functional thinking is based on three main aspects: the correspondence of an element of the definition set to exactly one element of the set of values; the covariation of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an object. This differentiation is in line with the developmental perspective on students' conceptualization of functions derived by Breidenbach et al. (1992) using the Action-Process-Object-Scheme (APOS) theory. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky and Wilson 2013).

Learning environments with experimentation activities have proven to be beneficial for functional thinking (Lichti and Roth 2018, Ganter 2013) and motivation (Ganter 2013). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view.

Furthermore, experiment activities enable constructivist settings, that lead to higher learning gains when using digital technologies (Drijvers et al. 2016) and provide embodied experiences, contributing as cognitive resources (Drijvers 2020).

Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and post-process (reflect results) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (ibid.), but a closer look reveals disparities: while hands-on material promotes the correspondence aspect and the association to the real situation, simulations foster covariational thinking, the interpretative usage of graphs and lead to higher overall gains in functional thinking (Lichti 2019).

The instrumental approach (Rabardel 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis (Artigue 2002) - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers 2020). In addition, embodied activities in a task seem to contribute to the instrumental genesis (ibid.). From the viewpoint of instrumental genesis, the results of Lichti (2019) can be interpreted as follows: when using simulations, schemes that develop are concerned with variation and transition, while measurement procedures of the hands-on material induce schemes that concentrate on values and conditions (ibid.). The students working with hands-on material associate their argumentation more often with the material, while the rationale of students using simulations frequently relates to the graph. Again, the instrumental genesis can explain these disparities: the hands-on material stimulates basic modelling schemes, relating the situation to mathematical description. Simulations already contain models of a situation and when used as multi-representational systems (Balacheff and Kaput 1997) illustrate connections between model and mathematical representation (e.g. graph and table) that evoke schemes for these representations and their transfer.

The study presented here attempts to make use of all these beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

SETTING 1: EXPERIMENTS WITH HANDS-ON MATERIAL AND SIMULATIONS

The learning environment is set in a story of two friends preparing to build a treehouse. The student activities are structured in five contexts (see below for details), each one laid out like a scientific experimentation process with preparation, experimentation and post-processing phase. Starting off with hands-on material to activate modelling schemes and enable embodied experience, students are asked to make assumptions about a pattern or relationship and on that basis, estimate values. During experimentation phase they take a series of measurements and data is recorded in a table within a simulation (GeoGebra). The simulation is designed in accordance to the hands-on material and provides the opportunity to create a graph concurrent with the context animation and to display the measurements of the hands-on material (and a corresponding trendline). This gives students the opportunity for systematic variation and parallel observation of the altering quantities, to induce schemes with a dynamic view and covariational thinking. Above, it facilitates the time consuming but little challenging representational switch from table to graph (Bossé et al. 2011). In the post-processing phase the students verify their measurements and analyse the graph (interpreting and interpolating). Subsequently they get back to the real material to check their estimations from preparation phase. Finally, they elaborate on the answer to the overarching task (calculate the amount of material needed to build the treehouse) based on the insights from experimentation activities, bringing together the modelling and representational schemes developed.

SETTING 2: ALTERNATIVE COMBINATION OF ARTEFACTS WITH A FOCUS ON COVARIATION

In setting 1 proposed above the measurement plays a dominant role, which sets a focus on the individual values of quantities and on single states of the relationship. This leads to a pointwise view of functions (Monk 1992), promotes the action concept and concentrates on the correspondence aspect (see above). In accordance with Breidenbach et al. (1992) and Dubinsky and Wilson (2013) it would be desirable to shift this focus to a process concept and to covariation, especially since possible sources of student' difficulties with functional relationships are seen in the dominance of numerical settings in school (Goldenberg et al. 1992). Together with the close relation of covariation to the difficult concept of variables (Leinhardt et al. 1990), this led to the call for a qualitative approach to functions (Thompson 1994; Falcade et al. 2007; Thompson et al. 2013) to facilitate the idea of covariation. Thus, in a second setting explicitly choses a non-numerical approach for experimenting with immediate examination of covariation.

The learning environment of setting 2 is structured accordingly to setting 1, with modifications in the experimental structure of the contexts: in the preparation phases of the first three contexts the students are only briefly introduced with estimation tasks based on hands-on material, before they use simulations to identify the related quantities. In the following experimental phase, the students observe the variation and covariation of the quantities in the simulations and verbally describe the relationships discovered. Subsequently graphs are generated within the simulations and in the post-processing phase students are asked to analyse the form of the graphs and connect their insights with the relationship described in the previous phase, before they observe individual values of quantities to check their estimations and answer the overarching task like in setting 1. The last two contexts are again briefly introduced with hands-on material and estimation tasks, followed by the request for verbal descriptions of the relationships. Based on these descriptions and on their insights from the previous contexts, students are asked to group the contexts by their kind of covariation. The students then continue with the experimentation phase and take measurements with the hands-on material and then proceed with simulations as in setting 1. In the post-processing phase students are now asked to verify their hypotheses on the relationships and their grouping with the graphs and tables from the experimental phase. Finally, students check their estimations and answer the overarching task like in setting 1.

CONTEXTS

Both settings use a treehouse building story with identical overarching tasks. The contexts are implemented with the same hands-on material (see figure 1 and 2) and simulations, but different components of the simulations are visible in the settings.

Figure 1: Hands-on material of the first three contexts in setting 1 and 2

The contexts are chosen to represent a linear and a quadratic relationship and one with varying change rate: the perimeter of a circular disc determined by its diameter, the number of cubes needed for a "staircase" determined by the number of steps and the fill height of a vessel determined by the volume of water filled into.

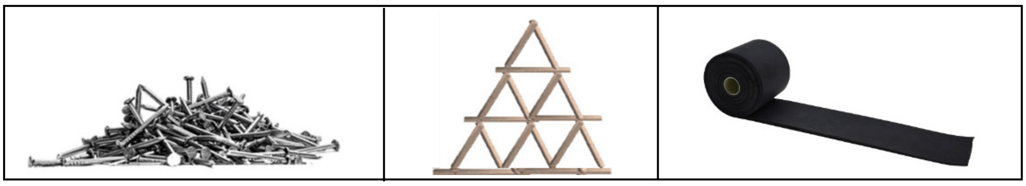

Figure 2: Hands-on material of contexts 4, 5 and bonus context 6

Contexts four and five (linear and quadratic) are the weight of a package of nails determined by the number of nails and number of beams needed for a woodwork determined by the number of floors. A bonus context for quick learners depicts the diameter of a unrolling tape determined by the length of tape that has been unrolled.

The simulations can be accessed in the digital classrooms (www.geogebra.org/classroom – for Setting 1, Team Engineers: Code HQX7 UZRQ – for Setting 2, Team Architects: Code D3XM DDSB).

STUDY DESIGN

A comparative intervention study (pre-post design) will contrast the two approaches to answer the following research question:

Is it possible to recognize differences between setting 1 and setting 2 in the processing of the tasks regarding the aspects of functional thinking and the conceptualization of function?

The intervention will take place at the University of Koblenz-Landau, as part of the mathematics laboratory program, where school classes work in groups of four in half-day projects with hands-on material and computer simulations. The intervention is preceded by a short test on functional thinking (FT-short, adapted from Lichti and Roth 2018), to compare the learning outcomes in both settings, and a three-minute intelligence screening (Baudson and Preckel 2016). Both tests take place approximately one week before. The intervention is designed for three 90-minutes-lessons including the post test of FT-short. A follow-up of the test is planned 4-8 weeks after the intervention. Two focus groups (low-/high-performer in FT-short) per school class will be videotaped. All student products and videos from the intervention are evaluated regarding the presence of the aspects of functional thinking (qualitative content analysis, validated category system from Lichti 2019) and the students' function conceptualization is assessed using the indicators from Dubinsky and Wilson (2013).

A pilot study intends to verify the comparability of the two approaches in terms of processing time and difficulty. Due to the corona shutdown and the ongoing rules for physical distancing the study is adapted to an online classroom supplemented with a "math box" containing the hands-on material. The dyadic approach and the videotaping in the pilot are replaced by an expert rating (questionnaire with four-point Likert scale N = 4 / open answers N = 9) and a reflective analysis based on the ALACT model (Korthagen 2017) with student teachers (N = 12, masters course in mathematics). The student pilot took place in two sessions with students from a high-school course held by the first author. They were assigned to the settings so that results in the FT pretest, overall math skills (half-term grade) and reading skills (half-term grade; two dyslexics) were equally represented in each setting.

PRELIMINARY RESULTS AND DISCUSSION

Here we present preliminary results of the student pilot study. One participant in each setting completed the whole program including the bonus context, two in setting 1 and three in setting 2 completed the 5th context (hence only bonus left) and the other students were still working on the 4th/5th context when time elapsed, so that regarding time both settings seem to be comparable. One identical task for both settings will be discussed in detail and compared with a related task of the FT pretest, indicating differences in the conceptual development.

In task no.48 students are asked to describe how the fill height of the liquid rises in the curved vessel using the graph and a given word list (slow, fast, steep, flat, rise, broad, narrow). In setting 2 all students were able to combine the fill height at least with the form of the vessel in their description, while in setting 1 only two students did connect their description of the fill height to the graph or the vessel. One student in setting 2 wrote:

> “The liquid rises slow first and from the value 1 on it gets faster because the vessel is broad at the beginning. From the value 2 on it becomes slower, because the vessel has a curve in the middle and was narrow but now becomes broader. Until value 6 the vessel keeps even, but from 6 on it rises up fast because the vessel becomes narrower and narrower. At point 10 there is a curve again, since the vessel is broader again.”

Although this student is not capable of interpreting the slope (or the form) of the graph, the references to the values indicate a connection between the measurement points in the graph and the fill height in the vessel. The description of the varying fill height shows a dynamic perspective and the concurrent statements about the variation in the vessel form show a covariational conception. One could argue that this conception does not include the fill volume itself, but the variation is given in the form of the vessel since liquid is filled with a constant rate.

Another student in setting 2 wrote:

> “When the line is steep in the graph, the water rises fast and the vessel is narrow. When the line is flat in the graph, the water rises slow and the vessel is broad.”

From a semantic view this student is arguing with conditions rather than changes, which reminds of a grading in intervals or chunky thinking as described by Castillo-Garsow et al. (2013). At the same time, he interprets the slope of the graph and connects it to the change in the fill height (representational switch), revealing a dynamic perspective, but the covariational conception does not include a dynamic perspective on the change of the form of the vessel (or at least it is not expressed).

The most elaborate description in setting 1 was:

> “At first it rises slow because the glass is broad, then it becomes narrower and narrower and it rises faster.”

Although this student grasps a variation in the fill height, the first argument points to the correspondence aspect and is based on simplification (constant diameter). The second statement shows a more dynamic view of the form of the vessel (“narrower and narrower”), but the related fill height is not described accordingly. Hence the covariational conception is only displayed in a preliminary stage.

These analyses only show an extract of the student documents, but they already indicate different stages of dynamic view and covariational conception in the two settings. To get an idea of the development towards these stages, we conclude with statements form a related task in the pretest. In this task students have to assign two out of four given fill curves to two vessels (cylinder and frustum of a cone). They provided the following explanations (comparably in both settings):

> “the vessel will fill faster and faster; the vessel is even, and the graph is straight; it becomes constantly more; at the bottom it fits more; first it takes a while because it is broad at the bottom”

Compared to the statements discussed above, one can detect a development in a) the connection between vessel, fill height and/or graph, b) in the dynamic view of one or more quantities and the graph and c) from a correspondence conception towards the variation of quantities and covariation.

Hence both settings are capable to foster functional thinking and the key aspect of covariation is addressed since students in both settings improved in the dynamic view and/or covariational thinking. The students in setting 2 seem to benefit more regarding the interpretation of the graph, which might be caused by the intensified usage of the multi-representational simulations. The qualitative approach of setting 2 might have set the focus on (co-)variation as intended, at least the improved dynamic perspective, better connection between quantities and rudiment covariational conceptions lead to this assumption. The main study will give more detailed insights to the development of the aspect of functional thinking and the conceptualization.

REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245–274.

Balacheff, N., & Kaput, J. J. (1997). Computer-based learning environments in mathematics. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 469–501). Dordrecht: Springer.

Baudson, T. G. & Preckel, F. (2016). mini-q: Intelligenzscreening in drei Minuten [mini-q: A Three-Minute Intelligence Screening]. *Diagnostica, 62*(3), 182–197.

Bossé, M., Adu-Gyamfi, K., & Cheetham, M. (2011). Assessing the Difficulty of Mathematical Translations: Synthesizing the Literature and Novel Findings. *International Electronic Journal of Mathematics Education, 6*(3), 113–133.

Castillo-Garsow, C., Johnson, H. L. & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243–1267.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y. & Maschietto, M. (2016). Uses of Technology in Lower Secondary Mathematics Education. *ICME-13 Topical Surveys*. Dordrecht: Springer

Drijvers, P. (2020). Embodied instrumentation: Combining different views on using digital technology in mathematics education. *Eleventh Congress of the European Society for Research in Mathematics Education*, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02436279

Dubinsky, E. & Wilson, R.T. (2013): High school students' understanding of the function concept. In: *The Journal of Mathematical Behavior, 32*(1), 83–101.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992): Dynamic representation and the development of an understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. (pp. 235–260). Washington, DC: Mathematical Association of America.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Lichti, M. (2019). *Funktionales Denken fördern: Experimentieren mit gegenständlichen Materialien oder Computer-Simulationen* [Fostering functional thinking: Experimenting with real materials or computer-based simulations]. Wiesbaden: Springer Spektrum.

Lichti, M., & Roth, J. (2018): How to Foster Functional Thinking in Learning Environments: Using Computer-Based Simulations or Real Materials. *Journal for STEM Education Research 1*(1-2), 148–172.

Monk, G. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–194). Washington, DC: Mathematical Association of America.

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments.* University of Paris 8, <hal-01020705>.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Vollrath, H.-J. (1986). Search strategies as indicators of functional thinking. *Educational Studies in Mathematics, 17*(4), 387–400.

Vollrath, H.-J. (1989). Funktionales Denken [Functional thinking]. *Journal für Mathematikdidaktik, 10*(1), 3–37.