INSPIRING LEARNING AND TEACHING OF FUNCTIONAL THINKING BY EXPERIMENTS WITH REAL AND DIGITAL MATERIALS

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Inspiring learning means to aim for promoting skills, knowledge, and interest. In a pre-post-test intervention study (N = 282, two experimental groups: real materials vs. digital materials, control group) we investigated whether experiments with real materials or digital materials based on GeoGebra are an appropriate measure to promote the functional thinking of sixth graders. Even though both types of material led to a significant increase in functional thinking, the increase of the digital material group was significantly higher. Digging deeper into data we found, that the use of either material has a different impact on the learning of functional thinking. To inspire teaching we use a concept of long-term in-service teacher training, were the analysis of video vignettes from student work processes within our online learning environment ViviAn is one measure among several others.

Keywords: teaching-learning laboratories, functional thinking, experiments, real and digital materials, video vignettes

1 TEACHING-LEARNING LABORATORIES INSPIRE LEARNING AND TEACHING

In German-speaking countries, teaching-learning laboratories (In German they are called “Lehr-Lern-Labore” (Priemer & Roth, 2019).) are getting more and more common when learning and teaching are to be inspired for students and pre- as well as in-service teachers at universities, especially in STEM-subjects. Research-based learning is a cornerstone of the work in teaching-learning laboratories. The Mathematics Laboratory “Maths is more” (cf. www.mathe-labor.uni-landau.de) at the Landau campus of the University of Koblenz-Landau comprises three closely linked pillars that are necessary components of every teaching-learning laboratory (cf. Roth 2019):

(1) It is initially a students’ laboratory in which entire school classes work in groups (four students each) within three 90 minutes slots on a curriculum topic in the sense of research-based learning (cf. Roth & Weigand 2014). On the basis of workbooks which contain written work instructions and in which the students record their procedures and results (cf. Roth, Schumacher & Sitter 2016, p. 195), the learners work independently with real and digital materials in learning environments according to Vollrath & Roth (2012, p. 151). The preparation and follow-up of the laboratory work take place in mathematics lessons at school and are supervised by the mathematics teacher.

(2) In addition, the Mathematics Laboratory “Maths is more” is a research laboratory to which (almost) all research activities of the working group Mathematics Education (secondary level) in Landau refer. This applies in one dimension from a pure perspective (Basic Science) with the purpose to understand the nature of mathematical thinking, teaching and learning to an applied perspective (Engineering) with the purpose to use such understandings to improve mathematics instruction (cf. Schoenfeld, 2000) and in another dimension from research on teaching mathematics in schools to university didactic research. A research project from this spectrum is presented in section 2.

(3) Last but not least, the Mathematics Laboratory “Maths is more” is also a teaching laboratory in which pre-service teachers apply and reflect their theoretical knowledge and skills in a practical way in the sense of research-based learning. They design learning environments for the students’ laboratory based on their knowledge of mathematics and mathematics education they gained at university

and support as well as diagnose students’ work on those learning environments. One means to interconnect practical work in the Mathematics Laboratory “Maths is more” to courses [1] for pre- and in-service teachers is the use of our self-developed video-tool ViviAn described in section 3.

2 INSPIRING LEARNING: EXPERIMENTS WITH REAL OR DIGITAL MATERIALS

Inspiring learning means to aim for promoting skills, knowledge, and interest of students. Especially in the case of topics that are relevant during the whole mathematics curriculum like functional thinking, it is important to inspire learning. Functional thinking can be described on the one hand by means of the three aspects mapping, covariation, and function as an object (Thomson 1994, Vollrath, 1989). On the other hand, functional thinking of students can be concluded if they are able to deal appropriately with forms of representation for functions like tables, graphs, formula, and situational descriptions and to change and translate between those representations (Nitsch, 2015). Experiments give the possibility to facilitate students’ understanding of functional relationships and foster their functional thinking by a scientific discovery process. This process includes generating hypotheses, testing hypotheses by performing experiments, and reflecting the results (De Jong 2005; Reid et al. 2003). Besides the use of real materials, studies also investigate and recommend the use of digital materials (Goldstone and Son 2005; Jaakkola et al. 2011). For an overview of arguments in favor of the use of real materials or digital materials respectively to foster functional thinking see Lichti and Roth (2018). Despite those existing arguments it has not been clear yet, whether (1) learning environments based on experiments with real materials or digital materials (GeoGebra) developed to foster the functional thinking of sixth-graders lead to significant effects on their functional thinking and (2) if those effects differ in both settings. Furthermore, we wanted to find out, if (3) learning environments based on experiments with real materials have a different effect on functional thinking than experiments with digital materials (GeoGebra). We decided to do an intervention study with students at the end of grade six as we wanted to be sure that it is our intervention, that eventually triggers the development in functional thinking. As in German mathematics curricula explicit teaching of functional relationships begins with grade seven, students would not have heard about it during mathematics lessons.

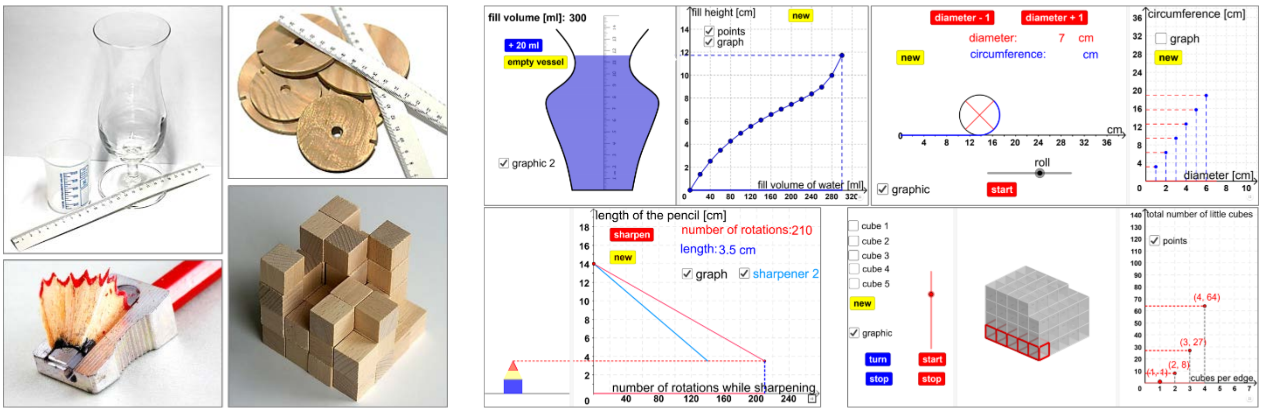

Figure 2.1: Four situations namely filling vessels, rolling circles, sharpening pencils, and building cubes, have been chosen for experiments with real materials (left) and digital materials (right)

To deal with the mentioned research questions four contexts have been chosen that could be used to perform experiments with real and digital materials respectively (cf. Fig. 2.1): (1) *Filling vessels* → relationship between the fill volume and the fill height of a vessel; (2) *rolling circles* → relationship between the diameter and circumference of a circle; (3) s*harpening pencils* → relationship between the number of rotations while sharpening a pencil and its remaining length; (4) *building cubes* → relationship between the number of little cubes fitting on the edge of a big cube and the number of little cubes needed to build the big cube. They cover different functional relationships, do not focus on linear functions only (de Beer et al. 2015) and offer the opportunity for the students to use the same actions and procedures when dealing with both types of material. This was essential, as two experimental groups namely *real material* (EG1) and *digital material* (EG2) had to be compared. Furthermore, they led to experiments that were practicable for students of grade 6 in both settings. It would for example not have been feasible for 30 students aged 11–12 years to experiment with burning candles at the same time. The complexity of the digital material also had to be appropriate for the students, as they should be able to use it intuitively.

After choosing contexts, we designed identical or at least equivalent tasks for both settings to guide the students through the experiments. We designed those tasks aiming to cover the three aspects of functional thinking according to Vollrath (1989) and using tables, graphs, and situational descriptions as forms of representation for functions. The syntactical form of representation (formula) was omitted, as students of grade 6 are not yet familiar with it.

The students were guided through the intervention using different types of tasks. We distinguished tasks for (i) estimating, (ii) experimenting, (iii) understanding the respective context, (iv) understand-ding the graphic form of representation, (v) applying (related to the results of the experiment) and (vi) transferring (tasks to the same context that go beyond the experiment). The tasks of the contexts at the beginning of the intervention focused on the task types (i)-(iv). The tasks on contexts of the second half of the intervention focused on types (v) and (vi). It was important to make the tasks in both settings equivalent or, if possible, identical. In addition, it had to be ensured that the students did not come to completely wrong conclusions. Therefore, there were help cards and solutions integrated into the sequence of tasks. The entire setting - the real materials, digital materials, and tasks - was tested for its usability and equivalence in a preliminary study. For this purpose, 30 students worked in groups on the tasks using the respective medium while they were being filmed. The video recordings were evaluated and based on the results the media and tasks were optimized.

For more information concerning the tasks as well as the way in which the digital materials were carefully constructed using GeoGebra (www.geogebra.org) to meet the requirements of the intervention study see Lichti and Roth (2018)

In a preliminary study, it was examined whether the real material, digital material, and tasks are suitable for students aged 11-12 and whether the processing time differs depending on the choice of medium. It turned out that the students need the same time to solve the tasks, regardless of the choice of medium. Based on the results of the preliminary study, an intervention study in a pre-post-control-group design was carried out shortly before the end of the school year 2015/16 ($N = 282$). 11 classes were involved in the intervention study. The students of each class were randomly assigned to the two experimental groups (EG1: $N = 111$; EG2: $N = 123$). Two further classes formed the control group (CG: $N = 48$). Figure 2.2 provides an overview of the intervention study.

A test to measure functional thinking was constructed and validated in which functional thinking was operationalized according to the three above named aspects, using tables, graphs, verbal descriptions as forms of representation in which functional relationships occur. The appropriate use of and change between these forms was considered an indication of functional thinking. After developing items to test functional thinking, their fit to the operationalization was controlled (expert rating: $N = 2$, $\kappa = 0.86$). Thereafter, a test was created and implemented among students at the age of 12 to 13 which is grade 7 in Germany (N = 221). Using item response theory, we controlled for Rasch scalability. Our test showed an expected a posteriori (EAP/PV) reliability based on plausible values of EAP/PV = 0.77. For further information on the test and the test-items see Lichti and Roth (2019).

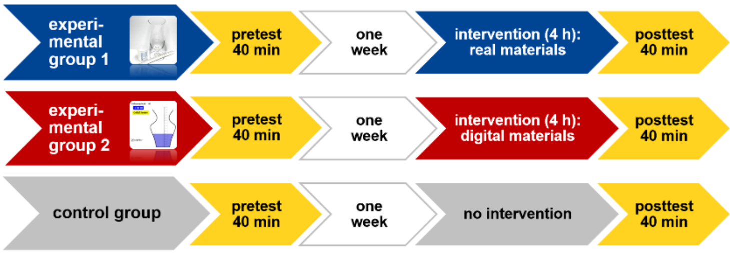

Figure 2.2: Setting of the intervention study: Students of 11 classes were randomly distributed on class level to two experimental groups, two further classes formed the control group (N = 282).

The students who participated in the intervention worked 4 school hours (45 minutes each) in individual work on equivalent tasks to functional contexts, only the material (real vs. digital) they used differed. Directly after the intervention, the post-test was processed and the data collected in the pre- and post-tests were evaluated using item-response theory. By means of virtual persons, the item difficulties were estimated, whereas the personal ability values were estimated by means of a 2-dim. Rasch model and 10 plausible values (Rost, 2004). A mixed ANOVA (between-Factor: Intervention, within-Factor: Time) (Field et al., 2013) was then calculated to compare functional thinking in pre-test and post-test.

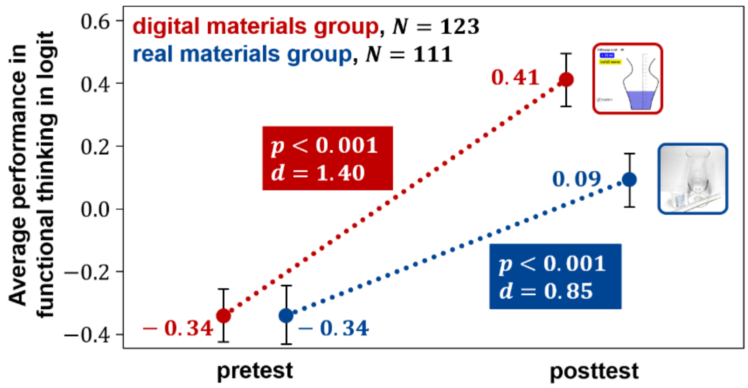

Figure 2.3: Mean test results in the functional thinking test presented in Logit (Effect size: Cohen's d)

The *control group* (N = 48) was analyzed with a paired Wilcoxon-Signed-Rank test. The analysis showed that the test without intervention did not have a significant effect on students' functional thinking (V = 423, p = .091, d = .26).

Experimental groups (EG): The mixed ANOVA resulted in two significant effects: First, there was a main effect of time $F(1, 11.79) = 36.90$, $p < .001$, $\eta_p^2 = .554$. Therefore, the functional thinking of the experimental groups considered as one group increased significantly with a large effect from $M = -.34$ logits (SD = .035) up to M = .25 logits (SD = .06). The pairwise t-test with Bonferroni correction led to the following results: The EGs did not differ before the intervention, but the functional thinking of both groups increased significantly from pre- to post-test (*real materials group*: $t(110) = -9.42$, $p < .001$, Cohen's $d = .85$; *digital materials group*: $t(122) = -16.46$, $p < .001$, Cohen's $d = 1.41$). Second, based on the mixed ANOVA there was an interaction effect between time and EG ($F(1, 25.820) = 8.856^{**}$, $p = .006$, $\eta_p^2 = .090$). In comparison, the increase in both groups from pre- to post-test is significantly different (cf. Fig. 2.3). The functional thinking improved significantly more in the digital material group (EG2) than in the real material group (EG2) with a medium effect ($\eta_p^2 = .09$)

To sum up, the functional thinking of students in grade 6 can be promoted using real and digital materials. Although both types of material generate a significant increase in functional thinking, the increase generated by digital materials is significantly higher. There seem to be differences in the way both types of material influence functional thinking. In order to understand the outcomes, a closer look at these differences is necessary. Consequently, the consideration of processes in functional thinking seemed appropriate. The following analysis of written processing results of students on tasks in functional contexts under consideration of the influence of real and digital materials is done to provide answers to this question.

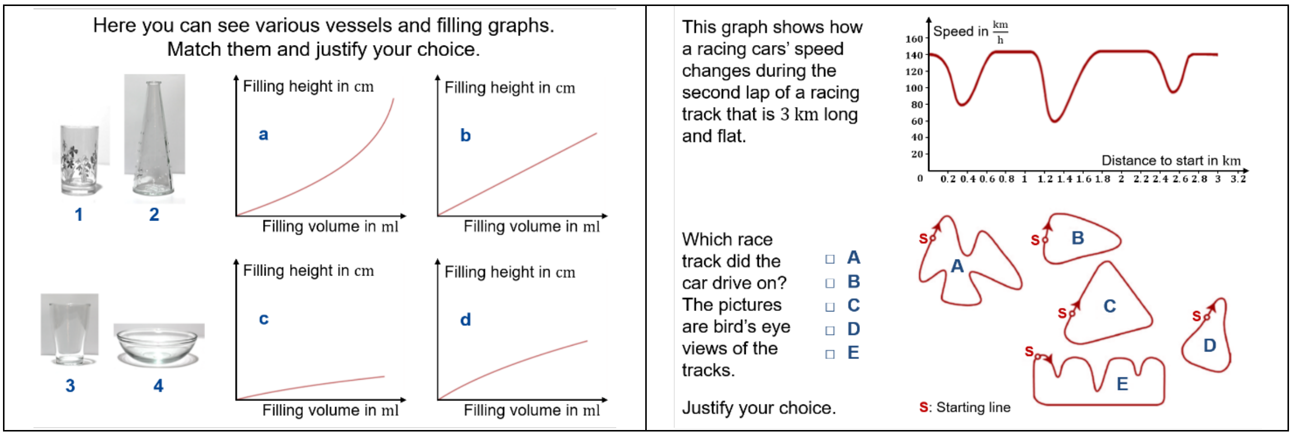

Figure 2.4: Task *filling vessels* (left) and task *racing car* (right)

The task *filling vessels* (cf. Fig. 2.4 left) was used during the intervention and the task *racing car* (cf. Fig. 2.3 right) was part of the follow-up test. *Filling vessels* requires the allocation of images of vessels (photos) to the corresponding filling graphs, *racing cars* the allocation of one of five race tracks to the corresponding speed graph. In both cases a depicted situation (photos, abstract representation of the race track from a bird's eye view) must be brought together with the respective graphic representation of the underlying functional context. In addition, the students had to justify their assignment. These justifications were examined by means of qualitative content analysis (Mayring, 2008), in which inductive categories were formed and confirmed by experts. Two category systems, one for each task, were created. Two raters used those category systems to code the students' justifications. Individual arguments could be provided with several categories. The interrater reliability was checked with Cohens Kappa (κ). The categories (1) shape of the vessel (κ = .97), (2) course of the graph (κ = .87), (3) state (κ = .97) and (4) change (κ = .93) were relevant for the task "filling vessels". While (1) and (2) reveal themselves in terms of content, in (3) the students had an actual state in view (the graph is steep.), in (4) they focused on a change (the vessel becomes narrower towards the top.) For the task of racing cars, the categories (1) speed and curves (κ = .92), speed (κ = .93), type of curves (κ = .87) and (4) graph-as-picture error of interpretation were used. (1) includes that the students argued with their everyday knowledge about the relationship between curves and speed (a car has to drive slower in curves), the categories (2) and (3) describe that the students worked with speed based on the graph or recognized that the track they were looking for had to have three different curves. (4) contains a typical misconception with the graph-as-picture error. This was followed by a χ^2 test comparison of category frequencies between students who had worked with real or digital materials to promote their functional thinking.

The comparison of the frequencies of the categories of the task *filling vessels* showed that students of the *real materials group* argued significantly more frequently with the shape of the vessel ($\chi^2 = 4.16$, $p = 0.04$) and states ($\chi^2 = 4.36$, $p = 0.04$). The students in the *digital materials group*, on the other hand, used the graph significantly more frequently in their argumentations ($\chi^2 = 6.62$, $p < 0.01$) and focused significantly more often on variation ($\chi^2 = 6.95$, $p = 0.008$).

When considering the task *racing car* we found that students in the *real materials group* chose the wrong solution significantly more frequently despite correct reference to their everyday knowledge about curves and speed ($p = 0.022$*). This could be due to problems interpreting the graph appropriately or difficulties with the representation of the race tracks. Tacking into account, that the graph-as-picture error did only occur in the *real materials group* suggests the assumption that problems with the interpretation of the graph are the cause for the difficulty in bringing everyday knowledge together with the task. The same was true for the category Speed ($p = 0.042$*). With regard to the type of curves category, it became clear that recognising the need for different types of curves seemed to be particularly important for the proper handling of the task. Out of 21 students who argued on the nature of the curves, 20 chose a correct solution. Also noteworthy was the appearance of the graph-as-picture error. The graph-as-picture error occurred in 36% of the students in the real materials group but by none of the students in the digital materials group. These findings led to a number of hypotheses regarding the processing of tasks on functional relationships. Three of these hypotheses focus on the aspects according to Vollrath. Thus it was assumed that (1) the real materials group had an advantage with regard to the promotion of the understanding of *mapping* since these students increasingly argued with states. In contrast, the digital materials group seemed to have an advantage in promoting the understanding of (2) *covariation* and (3) *function as object*. (2) resulted from the focus of the digital materials group on change and the course of the graph, (3) from the lower difficulties the students revealed in linking situation and graphical representation in the racing car task.

In order to verify the validity of these hypotheses, data from pre- and post-tests were used to measure the effectiveness of the intervention. For this purpose, the increases in the mean solution rates with regard to each test item were considered separately for the real materials group and the digital materials group. The items were grouped according to whether the real materials or digital materials group had achieved a larger increase in the mean solution rate. There were 9 items for the real materials group and 17 items for the digital materials group. These were distributed clearly in different ways between the groups with regard to the aspect function as object: 2 function-as-object items were found in the real materials group, 7 in the digital materials group. This was interpreted as a further indication for hypothesis (3).

An analysis of the items grouped in this way followed in order to identify commonalities between them that could explain the achievement of a larger increase in the respective group. It was noticeable that the 9 items that were allocated to the material group often dealt with the identification of pairs of values and thus indicated an advantage for the group with regard to the mapping aspect. The 17 items of the digital materials group, on the other hand, increasingly included anticipation and comparison of gradients and rates of change. This provided evidence for the hypotheses (1) and (2). It was also noticeable that the digital materials group achieved a larger increase in all items using tables. Perhaps, since students in this group only had to “click”, they had more free capacity to deal with the table as a form of representation. In contrast, the real material group achieved a larger increase in items of the cube context, which required a spatial understanding. Apparently, this kind of understanding could be promoted better by the use of real materials.

In summary, it must be stated that real materials and digital materials have different influences on the processing of tasks in functional contexts. There are indications that digital materials can improve the understanding of the aspects of covariation and function as object, while materials can improve the understanding of the mapping aspect. Furthermore, the work with certain contexts and the handling of forms of representation also seem to depend on the choice of the material used for the promotion. A comprehensive promotion of functional thinking should therefore include working with real materials as well as with digital materials to inspire learning, since both typs of material seem to complement each other perfectly.

3 INSPIRING TEACHING: VIDEO VIGNETTES FOR THE ANALYSIS OF TEACHING AND LEARNING PROCESSES

ViviAn is an acronym that stands for “**vi**deo **vi**gnettes for the **an**alysis of teaching processes”. The ViviAn learning environment (cf. www.vivian.uni-landau.de and Roth 2019) was developed because there was no diagnostic tool that met all the following requirements:

(1) Pre- and in-service teachers who process diagnostic tasks for video vignettes of group work processes of students should essentially have the same information on the situation presented as the supervising teacher of the group of students would have.
(2) Edits for diagnostic tasks are entered into a text field directly next to the corresponding tasks and automatically saved.
(3) After completing a diagnostic assignment individually, pre- and in-service teachers receive feedback from experts which they can compare to their own answers.
(4) Pre- and in-service teachers can carry out the diagnostic assignments at a time of their choice within a specified time window at a location of their choice.
(5) There is a user administration that allows lecturers to individually approve video vignettes (also via adjustable time intervals) and allows them to be used only after receipt of a signed data protection declaration.
(6) Lecturers can call up in an overview which video vignettes have already been processed by the individual pre- or in-service teacher in the course [1].
(7) All video data of the persons depicted are protected, stored on a server of the University of Koblenz-Landau under the complete control and exclusive administrative access of two administrators of the working group Didactics of Mathematics (secondary levels).

In particular, the first four points of the enumeration led to the development of the ViviAn learning environment and its interface. The functional scope of these is explained below.

Figure 3.1 presents the interface of the ViviAn learning environment. A video vignette is embedded in the center, showing an excerpt of approximately three minutes from an authentic group work phase of students in the Mathematics Laboratory “Maths is more”. The video player allows you to start, stop, fast-forward and rewind within the video vignette at any time. The group work process was recorded in bird’s eye view. The selected camera perspective supports the observation of the entire learning group as well as the focus on individual learners. Furthermore, all actions on the material are clearly visible. In phases in which the students essentially work with digital material, the screen recording (video of the students’ screen actions) is displayed at the center of the ViviAn learning environment. To ensure that the interaction of the learners at the table is not lost, the recording of the previously described camera perspective is displayed in reduced size in the lower-left corner of the video. In order to be able to clearly assign the verbalizations to the learners in the video, the person who is currently speaking was marked (cf. "S3" in Figure 3.1). This mark contains the identifier of the student. All students in the video vignettes were provided with one of the identifiers S1, S2, S3 or S4 starting from the bottom left clockwise in order to be able to identify them unambiguously for diagnostic tasks and answers.

Figure 3.1: Interface of the learning environment ViviAn (cf. www.vivian.uni-landau.de)

A window can be opened above the video vignette via the "*Learning environment: Topic and Goals*" button. In this window, the subject and learning objectives of the learning environment in which the students are working in the video are briefly displayed. This information should be called up at first for an initial overview. To the right of the video vignette in the "*Meta level*" box, pre- or in-service teachers can access information that a teacher in the classroom usually has. The "*Student Profiles*" button retrieves information about the learners in the video (including age, grade and type of school attended). Below this is the seating plan on which the learners (for clear communication) are numbered S1, S2, S3, S4 from left to right. This is meant to facilitate the overview of the events and access to individual students.

Below this is the button "*Temporal classification*". It opens the content plan of the three 90 minutes slots in which the students work in the students' laboratory. This enables the pre- or in-service teachers to see which learning objective is to be achieved by the task and which contents the learners are working on before and after the situation shown in the video vignette.

To the left of the video in the box "*Student level*", materials from the learning environment that learners work with or produce during the learning process can be retrieved. The "*Task*" button opens the task that the learning group processes in the video in a pop-up window (cf. Fig. 3.2). With the button "Materials" photos of the real material with which the learners work in the displayed situation can be called up in a pop-up window (cf. Fig. 3.2). If the students in the video use digital material it is shown directly in the pop-up window. The digital material, usually a GeoGebra-based simulation, has the full range of functions and can be used by the pre- or in-service teachers in the same way as the learners in the video vignette. In this way, their actions can be understood in the best possible way.

The "*Student documents*" button displays the written work results of all learners from the video. These can be selected via tabs and arranged so that any edits can be compared in pairs (cf. Fig. 3.3). This allows further access to the respective depth of reflection of the individual students.

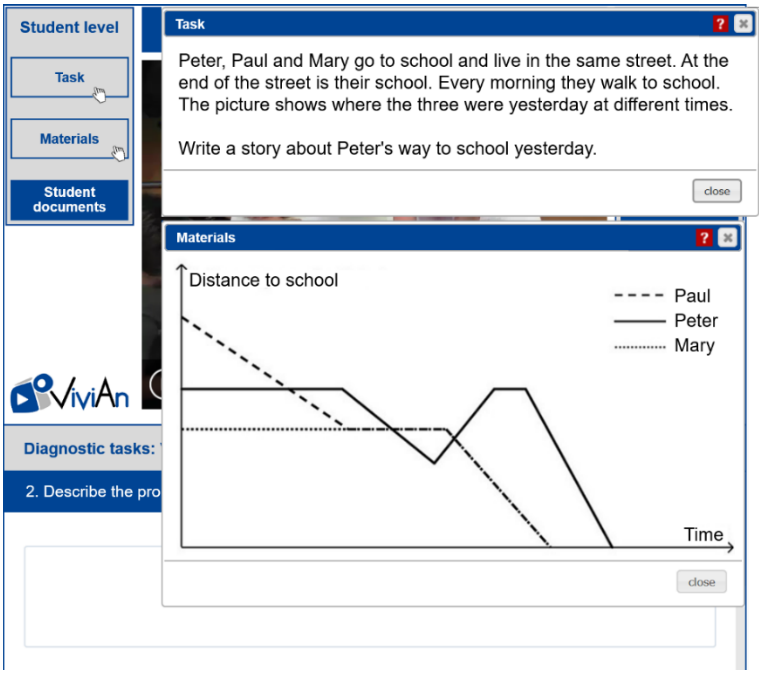

Figure 3.2: ViviAn-interface with open windows *Task* and *Materials*

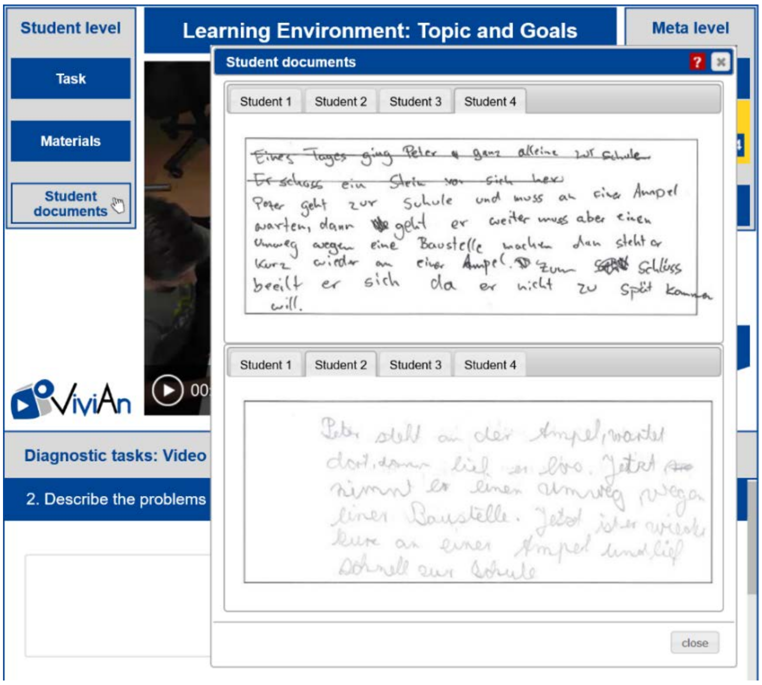

Figure 3.3. ViviAn-interface with open window *Student documents*

The “*Diagnostic tasks*” button at the bottom right of the ViviAn interface calls up the diagnostic tasks, which are then displayed individually below the video (cf. Fig. 3.4). The diagnostic assignments in the ViviAn learning environment each focus on a specific content aspect of mathematics learning in the situation presented in the video vignette, for which the necessary theoretical facets and pedagogical content knowledge were previously discussed in-depth in the course [1]. This shifting of the main focus of diagnosis to theoretical aspects previously addressed in the course [1] is intended to help pre- and in-service teachers become accustomed to carrying out their diagnoses on a theoretical basis. The diagnostic tasks are either open items that have to be answered in free text format or consist of a combination of a closed and an open item. The closed items are single-choice and multiple-choice questions. In order to encourage the pre- or in-service teachers to deal intensively and in detail with the situation, each closed item is followed by a question in free text format, which requires a justification of the previously chosen answer. Typical work assignments within ViviAn, which are intended to help train pre- and in-service teachers’ ability to diagnose group work situations of students (cf. Bartel & Roth, 2019), invite them to

- work on the tasks of the students,
- describe observations,
- interpret observations and give reasons for these interpretations (basic ideas, students’ ideas, etc.),
- propose and justify adaptive teaching.

If a pre- or in-service teacher has entered a text in the box for the answers to diagnostic tasks (see Fig. 3.4), answered the single-choice and multiple-choice questions, if applicable, and sent them by clicking on the box “*Next*” down on the page, a feedback page opens below the video (see Fig. 3.5).

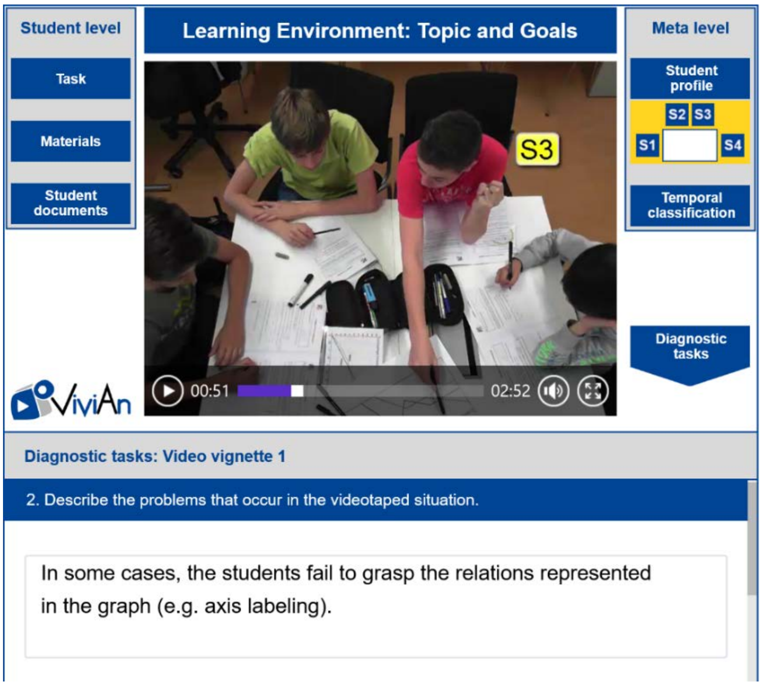

Figure 3.4: ViviAn-interface with open diagnostic task and a provided answer

The following is displayed on this page:

(1) The diagnostic task that the pre- or in-service teacher just completed,
(2) the text that the pre- or in-service teacher has just entered, as well as the single-choice or multiple-choice questions, if applicable, with markers for the selected answers,
(3) expert diagnoses as feedback on both closed and open questions. These are short texts containing analyses as well as corresponding justifications for the respective diagnostic assignments.

Figure 3.5. ViviAn-interface with feedback to an answer of a pre-service teacher

In this way, pre- and in-service teachers can compare their answers with expert answers and thus reflect on it. In two independent studies, Bartel and Roth (2017) and Enenkiel and Roth (in print) were able to show that the diagnostic abilities of teachers can be significantly improved in this way.

In order to enable feedback in the form of expert diagnoses, all video vignettes were first transcribed. The transcripts were then analyzed using mathematics education literature on the aspect of interest, e.g. learning the concept of function. Based on this and already completed processing of the videos, both by pre-service an in-service teachers in the context of a preliminary study and by researchers in mathematics education, an analysis with corresponding justifications was then carried out for each diagnostic task. Before being used in ViviAn, the correctness of this analysis was checked by at least three researchers in mathematics education and, if necessary, sharpened.

In this paper, different aspects of inspiring teaching and learning were addressed, starting with teaching-learning laboratories, continuing with the use of real and digital materials to learn functional thinking, and ending with the use of the learning environment ViviAn to train diagnostic competences of teachers to inspire their teaching. The inspiration can be manifold, but, as should be shown in the article, it has to meet various criteria in order to be successful.

NOTES

1. A *course* is in the case of pre-service teachers a lecture on mathematics education, in the case of in-service teachers a teacher training course.

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